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Quantization of three-wave equations

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Abstract

The subject of this paper is the revision of discretization and quantization of two similar classical integrable *Hamiltonian* systems in three-dimensional spacetime: the standard three-wave equations and the 'modified' three-wave equations (the Darboux–Manakov–Zakharov system in two distinguished gauges). The quantized systems in discrete spacetime may be understood as the regularized integrable quantum field theories. Integrability of the theories, and in particular the quantum tetrahedron equations for vertex operators, follow from the quantum auxiliary linear problems. The principal object of the lattice field theories is the Heisenberg discrete-time evolution operator constructed with the help of vertex operators.

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1. Introduction

The aim of this paper is the formulation of integrable quantum field theories in (2 + 1)-dimensional spacetime corresponding to the three-wave equations or, more generally, the Darboux–Manakov–Zakharov system. The subject of quantization is a Hamiltonian structure of classical equations of motion. The general Darboux–Manakov–Zakharov system in three-dimensional spacetime has two (inequivalent) Hamiltonian structures with ultra-local Poisson brackets for canonical fields providing two classes of three-dimensional quantum integrable theories. The purpose of the paper is somewhat twofold.

1. Three-dimensional quantum integrable systems, usually associated with Zamolodchikov's quantum tetrahedron equation, are viewed in this paper as quantized classical discrete three-wave/Darboux–Manakov–Zakharov systems. The three-wave [1] and more general Darboux–Manakov–Zakharov [2] equations and their discretization [3, 4] were the subject of intensive study in the framework of multidimensional integrable discrete geometry during the past decade. It is impossible to cite all relevant publications (see, e.g., [5–9]).

From the other side, two classes of solutions of the quantum tetrahedron equation (three-dimensional extension of the Yang–Baxter equation) [10–14] and related three-dimensional modifications of the quantum inverse scattering method unravel the hidden three-dimensional structure of affine quantum algebras (actually, two classes of solutions if the tetrahedron equation reproduces the quantum groups together with their cyclic and highest weighs evaluation representation theory). It is impossible to cite all publications relevant to quantum groups and QISM either. In particular, the basic quantum integrable models such as XXZ spin chain, Toda chain, chiral Potts model, quantum sine-Gordon and Liouville models, one-dimensional Bose gas, etc are the results of a minimal 3d $\to$ 2d 'compactification' of three-dimensional quantum systems with different types of 3d lattices (few details of such dimension-rank transmutation may be found in [11, 13–15]). This paper is an attempt to manifest the conceptual integrity of all modern mathematical physics, from the discrete geometry to the quantum integrable systems.

2. The second quantum-field-theoretical motif of the paper is the following. Given a classical theory with action $S[\phi]$, there is the universal prescription for the quantization. The amplitude between a state $\phi_{\text{in}}(\mathbf{r})$ at time t_1 and a state $\phi_{\text{out}}(\mathbf{r})$ at time t_2 is

$$\langle\phi_{\text{out}}|\phi_{\text{in}}\rangle = \int_{\substack{\phi(\mathbf{r},t_1)=\phi_{\text{in}}(\mathbf{r})\\ \phi(\mathbf{r},t_2)=\phi_{\text{out}}(\mathbf{r})}} \mathscr{D}\phi\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\phi]}. \tag{1}$$

The perturbation-theory approach to the definition of $\mathscr{D}\phi$ we reject from every beginning. The fundamental way to define the measure in the Feynman integral is the discretization of the spacetime. The measure is to be understood as

$$\mathscr{D}\phi = \lim \prod_{i,j} \mathrm{d}\phi(\mathbf{r}_i, t_j), \tag{2}$$

where the limit symbol stands for infinitely dense discretization of the spacetime $(\mathbf{r}, t)$.

Thus, the discretization of classical system is the first step towards the quantization. Only then may we look for a self-consistent Heisenberg quantum mechanics on the lattice and regard it as the regularized field theory.

A schematic outlook of milestones of the quantization method is the following. A classical theory is defined by the time dynamics for a field A, $\frac{\mathrm{d}}{\mathrm{d}t}A = f[A]$ (space-like degrees of freedom are omitted for brevity). The dynamics is generated by a Hamiltonian, $f[A] = \{H, A\}$, where $\{\,,\}$ is a properly defined Poisson bracket. The corresponding discrete-time dynamics is the evolution transformation $A(t + \Delta t) = F[A(t)]$. For brevity, we choose the scale $\Delta t = 1$. The key point is that making the discretization, we have to take care of the Hamiltonian structure of the dynamics. The discrete-time evolution must be a canonical transformation; it should preserve properly defined Poisson algebra of observable fields on space-like lattice. The quantum algebra of observables is a result of Dirac quantization of the Poisson algebra. The quantum evolution map $A(t) \to A(t+1)$ must be the automorphism of the algebra of observables allowing one to define the Heisenberg evolution operator,

$$A(t + 1) = UA(t)U^{-1}. \tag{3}$$

Such scheme is the algebraic realization of the discrete measure (2). Operator U acts in a representation space of the algebra of observables; the amplitude (1) in the Heisenberg form is

$$\langle\phi_{\text{out}}|U^N|\phi_{\text{in}}\rangle = \int_{\substack{\phi_0=\phi_{\text{in}}\\ \phi_N=\phi_{\text{out}}}} \underbrace{\prod_{n=1}^{N-1} \mathrm{d}\phi_n}_{\mathscr{D}\phi} \underbrace{\prod_{n=1}^{N} \langle\phi_n|U|\phi_{n-1}\rangle}_{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\phi]}}. \tag{4}$$

Thus, the principal object of quantum theory in discrete spacetime is not a local Hamiltonian but the one-step evolution operator. Also, the one-step evolution operator is not a transfer matrix.

In what follows, we revise the discretization of the spacetime serving the discretization of the Cauchy problem $\frac{\mathrm{d}}{\mathrm{d}t}A = f[A] \to A(t+1) = F[A(t)]$, construction of the quantum algebra of observables and the basis-invariant definition of Heisenberg evolution operator for both Hamiltonian structures of the Darboux–Manakov–Zakharov system.

2. Three-wave equations and Hamiltonian structures

We commence with a short reminding of the classical three-wave equations in three-dimensional space. In what follows, we use the short notations for the indices,

$$(i, j, k) = \text{any permutation of } (1, 2, 3). \tag{5}$$

2.1. Standard three-wave equations

The linear problem for the standard three-wave equations is the set of six differential relations

$$\partial_i \psi_j = A_{ij}\psi_i, \qquad i \neq j \tag{6}$$

for three auxiliary fields ψ_i. Consistency of (6) gives the equations for the six fields A_{ij}:

$$\partial_i A_{jk} = A_{ji}A_{ik}. \tag{7}$$

These are the equations of motion for a three-wave resonant system [1, 16] or the three-wave equations for the shortness.

2.2. 'Modified' three-wave equations

The three-wave system is related to the Manakov–Zakharov auxiliary linear problem

$$(\partial_i\partial_j - A_{ij}\partial_j - A_{ji}\partial_i + C_{ij})\phi = 0 \tag{8}$$

for the 'gauge' $C_{ij} = 0$ by means of non-local change of variables. There is another distinguished 'gauge' $C_{ij} = A_{ij}A_{ji}$ [17]. The consistency conditions for the auxiliary problem

$$(\partial_i\partial_j - A_{ij}\partial_j - A_{ji}\partial_i + A_{ij}A_{ji})\phi = 0, \qquad i \neq j \tag{9}$$

give similar to (7) equations for the fields A_{ij},

$$\partial_i A_{jk} = (A_{ij} - A_{ik})(A_{jk} - A_{ji}). \tag{10}$$

We call them the 'modified' three-wave equations.

2.3. Hamiltonian structure

Distinguishing property of equations (7) and (10) is that they are Hamiltonian equations with ultra-local Poisson brackets [16, 17].

It is convenient to use the single alphabetical indices instead of the numerical pairs. In this paper we will use the following convention for both systems:

$$\begin{array}{lll} A_{12} = A_a, & A_{13} = A_b, & A_{23} = A_c, \\ A_{21} = A_a^*, & A_{31} = A_b^*, & A_{32} = A_c^*. \end{array} \tag{11}$$

Note, our notations are not cyclic with respect to (1, 2, 3).

Equations (7) and (10) in three dimensions are extremum conditions for the local action

$$\mathcal{S} = \int d^3x (A_a^* \partial_3 A_a - A_b^* \partial_2 A_b + A_c^* \partial_1 A_c - V), \tag{12}$$

where

$$V = A_a^* A_b A_c^* - A_a A_b^* A_c \tag{13}$$

for the standard three-wave equations (7) and

$$V = (A_a - A_b)(A_a^* - A_c)(A_b^* - A_c^*) \tag{14}$$

for the modified three-wave equations (10). For a moment, we ignore the reality conditions; the star in all these notations does not mean the complex conjugation.

The time derivative ∂_t and space derivatives ∂_x, ∂_y for action (12) may be chosen by

$$\partial_1 = \partial_t - \partial_x, \qquad \partial_2 = -\partial_t, \qquad \partial_3 = \partial_t - \partial_y. \tag{15}$$

Let $\mathbf{r} = (x, y)$ stand for the space-like vector. The choice of the time direction (15) provides the Hamiltonians

$$\mathcal{H} = \int \mathrm{d}^2\mathbf{r} (A_a^* \partial_y A_a + A_c^* \partial_x A_c + V), \tag{16}$$

so that the equations of motion (7) and (10) are

$$\frac{\mathrm{d}}{\mathrm{d}t} A_a(\mathbf{r}, t) = \{\mathcal{H}, A_a(\mathbf{r}, t)\}, \qquad \text{etc}, \tag{17}$$

where the ultra-local Poisson brackets are defined by

$$\{A_v^*(\mathbf{r}', t), A_v(\mathbf{r}, t)\} = \delta(\mathbf{r}' - \mathbf{r}), \qquad v = a, b, c. \tag{18}$$

Any other same-time bracket is zero. Six fields A_v, A_v^* are thus the canonical variables for the phase space of both Hamiltonian systems. Three logarithmic potentials of the Darboux–Manakov–Zakharov system play the role of Lagrangian variables; they are not suitable for the quantization procedure.

Note finally, the 'gauge' transform $\phi \to \mathrm{e}^u \phi$ of (8) especially with u depending on canonical variables is not a gauge transform of a Hamiltonian system.

Relations (11) and (15) are just one of many possible conventions for the field notations and spacetime separation. All such conventions in the continuous case are equivalent. The choice (11) and the signs in (15) fix properly the time direction and space-like surface for the discretization of the Cauchy problem.

3. The discretization

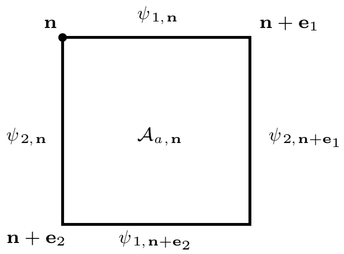
Now we proceed to the discrete analogue [3, 4] of the standard and modified three-wave equations (7) and (10). The straightforward discretization of three-dimensional space gives the cubic lattice

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \qquad x_i \in \mathbb{R} \quad \mapsto \quad \mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3, \qquad n_i \in \mathbb{Z} \tag{19}$$

The discrete analogue of derivative is the difference derivative,

$$\partial_i \mapsto \Delta_i, \qquad \Delta_i \phi(\mathbf{n}) \stackrel{\text{def}}{=} \phi(\mathbf{n} + \mathbf{e}_i) - \phi(\mathbf{n}). \tag{20}$$

The straightforward discretization is applied to the linear problems (6) and (9), the discrete equations of motion appear as the consistency conditions.



$$\Leftrightarrow \qquad \begin{aligned} \psi_{1,\mathbf{n}+\mathbf{e}_2} &= \psi_{1,\mathbf{n}} + A_{21}(\mathbf{n})\psi_{2,\mathbf{n}} \\ \psi_{2,\mathbf{n}+\mathbf{e}_1} &= \psi_{2,\mathbf{n}} + A_{12}(\mathbf{n})\psi_{1,\mathbf{n}} \end{aligned}$$

Figure 1. Graphical representation of the elements of the linear equations (21): linear variables $\psi_{i,\mathbf{n}}$ are associated with the edges of cubic lattice and the fields $\mathcal{A}_{a,\mathbf{n}} = (A_{12}(\mathbf{n}), A_{21}(\mathbf{n}))$ are associated with the face of cubic lattice.

3.1. *Standard three-wave equations*

The discrete linear problem corresponding to (6),

$$\Delta_i \psi_{j,\mathbf{n}} = A_{ij}(\mathbf{n})\psi_{i,\mathbf{n}}, \tag{21}$$

provides the discrete equations of motion

$$A_{jk}(\mathbf{n}+\mathbf{e}_i) = \frac{A_{jk}(\mathbf{n}) + A_{ji}(\mathbf{n})A_{ik}(\mathbf{n})}{1 - A_{ji}(\mathbf{n})A_{ij}(\mathbf{n})}. \tag{22}$$

3.2. *'Modified' three-wave equations*

The discrete linear problem corresponding to (9),

$$\phi_{\mathbf{n}+\mathbf{e}_i+\mathbf{e}_j} - Q_{ji}(\mathbf{n})\phi_{\mathbf{n}+\mathbf{e}_i} - Q_{ij}(\mathbf{n})\phi_{\mathbf{n}+\mathbf{e}_j} + Q_{ij}(\mathbf{n})Q_{ji}(\mathbf{n})\phi_{\mathbf{n}} = 0, \tag{23}$$

where

$$Q_{ij}(\mathbf{n}) = 1 + A_{ij}(\mathbf{n}), \qquad \text{etc} \tag{24}$$

provides

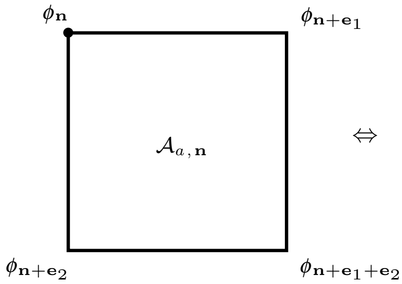
$$Q_{jk}(\mathbf{n}+\mathbf{e}_i) = \frac{Q_{ji}(\mathbf{n})Q_{ik}(\mathbf{n}) + Q_{ij}(\mathbf{n})Q_{jk}(\mathbf{n}) - Q_{ij}(\mathbf{n})Q_{ji}(\mathbf{n})}{Q_{ik}(\mathbf{n})}. \tag{25}$$

4. Constant time surface and evolution

Discrete equations of motion (22) and (25) evidently define a sort of one-step evolution: the fields at point $\mathbf{n}+\mathbf{e}_i$ are expressed in terms of the fields at point $\mathbf{n}$. However, an explicit form of a space-like discrete surface and detailed definition of a discrete-time corresponding to our choice (15) needs some discussion; the correct separation of space and time is the principal part of dynamics.

The key point is the geometrical structure of our discretization and the geometrical structure of the linear problems (21) and (23).

Let vector $\mathbf{n}$ in (19) stand for the vertex of the cubic lattice. The cubic lattice consists of *vertices* $\mathbf{n}$, *edges* $(\mathbf{n}, \mathbf{n}+\mathbf{e}_i)$ and *faces* $(\mathbf{n}, \mathbf{n}+\mathbf{e}_i, \mathbf{n}+\mathbf{e}_j, \mathbf{n}+\mathbf{e}_i+\mathbf{e}_j)$. The linear variable $\psi_{i,\mathbf{n}}$ of (21) should be associated with $(\mathbf{n}, \mathbf{n}+\mathbf{e}_i)$-edge, the linear variable $\phi_{\mathbf{n}}$ should be associated with $\mathbf{n}$-vertex and the fields $A_{ij}(\mathbf{n})$, $A_{ji}(\mathbf{n})$ should be associated with $(\mathbf{n}, \mathbf{n}+\mathbf{e}_i, \mathbf{n}+\mathbf{e}_j, \mathbf{n}+\mathbf{e}_i+\mathbf{e}_j)$-face. This is justified by the structure of the linear equations, see figures 1 and 2. In what follows, symbol $\mathcal{A}_{v,\mathbf{n}}$ stands for the collection of fields on $(v, \mathbf{n})$ face.



$$\Leftrightarrow \quad \phi_{n+\mathbf{e}_1+\mathbf{e}_2} - Q_{21}\phi_{\mathbf{n}+\mathbf{e}_1} - Q_{12}\phi_{\mathbf{n}+\mathbf{e}_2} + Q_{12}Q_{21}\phi_{\mathbf{n}} = 0$$

Figure 2. Graphical representation of the elements of the linear equation (23): linear variables $\phi_{\mathbf{n}}$ are associated with the vertex of cubic lattice and the fields $\mathcal{A}_{a,\mathbf{n}} = (Q_{12}(\mathbf{n}), Q_{21}(\mathbf{n}))$ are associated with the face of cubic lattice.

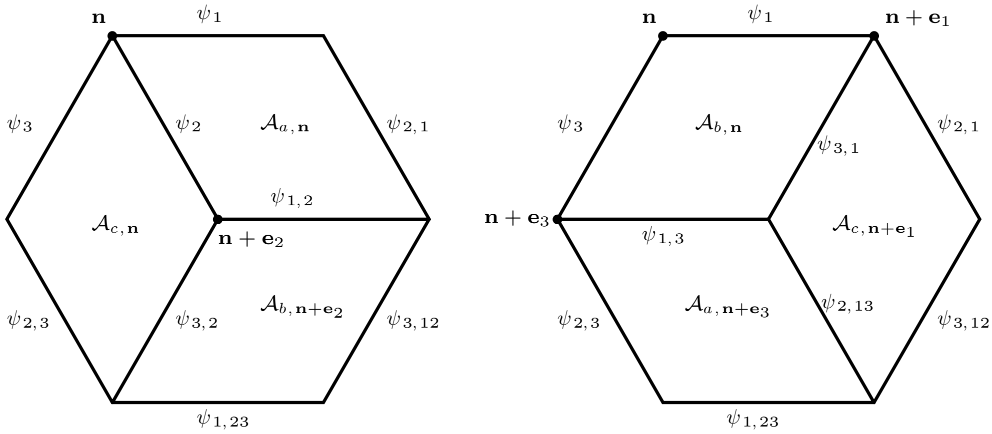


Figure 3. Consistency around the cube for the linear problem of figure 1. Here $\psi_1 = \psi_{1,\mathbf{n}}$, $\psi_{1,2} = \psi_{1,\mathbf{n}+\mathbf{e}_2}$, $\psi_{1,23} = \psi_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}$, etc. The left-hand side corresponds to equations (26), the right-hand side corresponds to equations (27); $\mathcal{A}_{a,\mathbf{n}} = (A_{12}(\mathbf{n}), A_{21}(\mathbf{n}))$, etc, according to (11).

Take now up the auxiliary linear problem (21) and more detailed derivation of (22). There are two ways to express $\psi_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}$, $\psi_{2,\mathbf{n}+\mathbf{e}_3}$, $\psi_{3,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}$ in terms of $\psi_{1,\mathbf{n}}$, $\psi_{2,\mathbf{n}+\mathbf{e}_1}$, $\psi_{3,\mathbf{n}}$. The one way is to use the six relations,

$$\begin{aligned} &\Delta_3\psi_{1,\mathbf{n}+\mathbf{e}_2} = A_{31}(\mathbf{n}+\mathbf{e}_2)\psi_{3,\mathbf{n}+\mathbf{e}_2}, \quad \Delta_3\psi_{2,\mathbf{n}} = A_{32}(\mathbf{n})\psi_{3,\mathbf{n}}, \quad \Delta_2\psi_{1,\mathbf{n}} = A_{21}(\mathbf{n})\psi_{2,\mathbf{n}}, \\ &\Delta_1\psi_{3,\mathbf{n}+\mathbf{e}_2} = A_{13}(\mathbf{n}+\mathbf{e}_2)\psi_{1,\mathbf{n}+\mathbf{e}_2}, \quad \Delta_2\psi_{3,\mathbf{n}} = A_{23}(\mathbf{n})\psi_{2,\mathbf{n}}, \quad \Delta_1\psi_{2,\mathbf{n}} = A_{12}(\mathbf{n})\psi_{1,\mathbf{n}}. \end{aligned} \tag{26}$$

The second way is to use the six other relations,

$$\begin{aligned} &\Delta_3\psi_{2,\mathbf{n}+\mathbf{e}_1} = A_{32}(\mathbf{n}+\mathbf{e}_1)\psi_{3,\mathbf{n}+\mathbf{e}_1}, \quad \Delta_2\psi_{1,\mathbf{n}+\mathbf{e}_3} = A_{21}(\mathbf{n}+\mathbf{e}_3)\psi_{2,\mathbf{n}+\mathbf{e}_3}, \quad \Delta_3\psi_{1,\mathbf{n}} = A_{31}(\mathbf{n})\psi_{3,\mathbf{n}}, \\ &\Delta_2\psi_{3,\mathbf{n}+\mathbf{e}_1} = A_{23}(\mathbf{n}+\mathbf{e}_1)\psi_{2,\mathbf{n}+\mathbf{e}_1}, \quad \Delta_1\psi_{2,\mathbf{n}+\mathbf{e}_3} = A_{12}(\mathbf{n}+\mathbf{e}_3)\psi_{1,\mathbf{n}+\mathbf{e}_3}, \quad \Delta_1\psi_{3,\mathbf{n}} = A_{13}(\mathbf{n})\psi_{1,\mathbf{n}}. \end{aligned} \tag{27}$$

Graphical representation for (26) and (27) is the left and right-hand sides of figure 3. The collection of relations (26) and (27) corresponds to the faces of the cube $(\mathbf{n}, \mathbf{n}+\mathbf{e}_i, \mathbf{n}+\mathbf{e}_i+\mathbf{e}_j, \mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$—the discrete spacetime consistency is the consistency around the cube.

The way to introduce the discrete time is to identify the discrete spacetime fields on the left-hand side of figure 3 with the discrete-time t, and identify the discrete spacetime fields on

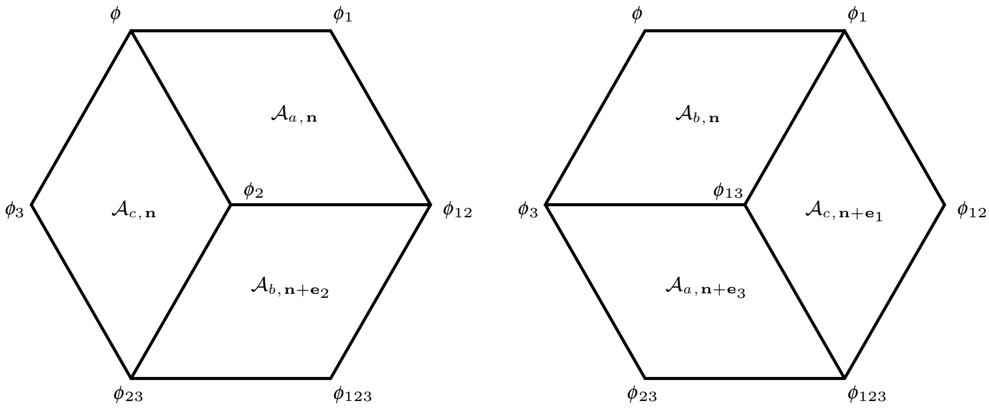


Figure 4. Consistency around the cube for the linear problem of figure 2. Here $\phi = \phi_{\mathbf{n}}, \phi_1 = \psi_{\mathbf{n}+\mathbf{e}_1}, \phi_{12} = \phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}$, etc. Convention (11) is taken into account, $\mathcal{A}_{a,\mathbf{n}} = (Q_{12}(\mathbf{n}), Q_{21}(\mathbf{n}))$, etc.

the right-hand side of figure 3 with the discrete-time $t + 1$. This exactly corresponds to (15)

$$\mathbf{e}_1 = \mathbf{e}_t - \mathbf{e}_x, \qquad \mathbf{e}_2 = -\mathbf{e}_t, \qquad \mathbf{e}_3 = \mathbf{e}_t - \mathbf{e}_y, \tag{28}$$

and therefore

$$\mathbf{n} = t\mathbf{e}_t + x\mathbf{e}_x + y\mathbf{e}_y : \qquad x = -n_1, \qquad y = -n_3, \qquad t = n_1 - n_2 + n_3. \tag{29}$$

The following table gives the correspondence between the initial $A_{ij}(\mathbf{n})$—notations and the spacetime notations, convention (11) is taken into account:

$$\begin{aligned} &A_{12}(\mathbf{n}) = A_a(x,y,t) = A_a, && A_{21}(\mathbf{n}) = A_a^*(x,y,t) = A_a^*,\\ &A_{13}(\mathbf{n}+\mathbf{e}_2) = A_b(x,y,t) = A_b, && A_{31}(\mathbf{n}+\mathbf{e}_2) = A_b^*(x,y,t) = A_b^*,\\ &A_{23}(\mathbf{n}) = A_c(x,y,t) = A_c, && A_{32}(\mathbf{n}) = A_c^*(x,y,t) = A_c^*. \end{aligned} \tag{30}$$

Here the third columns are the shortened notations. The right-hand side of figure 3 implies thus

$$\begin{aligned} &A_{12}(\mathbf{n}+\mathbf{e}_3) = A_a(x,y-1,t+1) = \overline{A}_a, && A_{21}(\mathbf{n}+\mathbf{e}_3) = A_a^*(x,y-1,t+1) = \overline{A}_a^*,\\ &A_{13}(\mathbf{n}) = A_b(x,y,t+1) = \overline{A}_b, && A_{31}(\mathbf{n}) = A_b^*(x,y,t) = \overline{A}_b^*,\\ &A_{23}(\mathbf{n}+\mathbf{e}_1) = A_c(x-1,y,t+1) = \overline{A}_c, && A_{32}(\mathbf{n}+\mathbf{e}_1) = A_c^*(x-1,y,t+1) = \overline{A}_c^*, \end{aligned} \tag{31}$$

Here the third columns are the shortened notations as well. The consistency condition of (26) and (27), completely equivalent to equation (22), may be rewritten in shortened notations as

$$\begin{cases} \overline{A}_a^* = A_a^* + A_b^* A_c, & \overline{A}_a = \dfrac{A_a(1 - A_c A_c^*) + A_b A_c^*(1 - A_a A_a^*)}{1 - \overline{A}_b \overline{A}_b^*},\\ \overline{A}_b^* = A_b^*(1 - A_c A_c^*) - A_a^* A_c^*, & \overline{A}_b = A_b(1 - A_a A_a^*) - A_a A_c,\\ \overline{A}_c^* = \dfrac{A_c^*(1 - A_a A_a^*) + A_a A_b^*(1 - A_c A_c^*)}{1 - \overline{A}_b \overline{A}_c^*}, & \overline{A}_c = A_c + A_a^* A_b. \end{cases} \tag{32}$$

In the absolutely similar way one may consider the discrete linear problem of figure 2. The corresponding picture is figure 4.

Introducing the same-time variables for the left-hand side of figure 4 analogously to (30),

$$\begin{array}{ll} Q_{12}(\mathbf{n}) = u_a(x,y,t) = u_a, & Q_{21}(\mathbf{n}) = w_a(x,y,t) = w_a, \\ Q_{13}(\mathbf{n}+\mathbf{e}_2) = u_b(x,y,t) = u_b, & Q_{31}(\mathbf{n}+\mathbf{e}_2) = w_b(x,y,t) = w_b, \\ Q_{23}(\mathbf{n}) = u_c(x,y,t) = u_c, & Q_{32}(\mathbf{n}) = w_c(x,y,t) = w_c \end{array} \tag{33}$$

and for right-hand side of figure 4 analogously to (31),

$$\begin{array}{ll} Q_{12}(\mathbf{n}+\mathbf{e}_3) = u_a(x,y-1,t+1) = \overline{u}_a, & Q_{21}(\mathbf{n}+\mathbf{e}_3) = w_a(x,y-1,t+1) = \overline{w}_a, \\ Q_{13}(\mathbf{n}) = u_b(x,y,t+1) = \overline{u}_b, & Q_{31}(\mathbf{n}) = w_b(x,y,t+1) = \overline{w}_b, \\ Q_{23}(\mathbf{n}+\mathbf{e}_1) = u_c(x-1,y,t+1) = \overline{u}_c, & Q_{32}(\mathbf{n}+\mathbf{e}_1) = w_c(x-1,y,t+1) = \overline{w}_c, \end{array} \tag{34}$$

we get the consistency condition in the shortened notations:

$$\begin{cases} \overline{u}_a = \dfrac{u_a w_b + u_b w_c - u_b w_b}{w_c}, & \overline{w}_a = \dfrac{w_a w_b u_c}{u_c w_c + w_a w_b - w_a w_c}, \\ \overline{u}_b = \dfrac{u_a w_a + u_b u_c - u_a u_c}{w_a}, & \overline{w}_b = \dfrac{u_c w_c + w_a w_b - w_a w_c}{u_c}, \\ \overline{u}_c = \dfrac{w_a u_b u_c}{u_a w_a + u_b u_c - u_a u_c}, & \overline{w}_c = \dfrac{u_a w_b + u_b w_c - u_b w_b}{u_a}. \end{cases} \tag{35}$$

These relations are completely equivalent to equation (25).

The derivation of the equations of motion as the consistency of linear problem around the cube is somewhat similar to the consistency approach to the integrable equations on quad-graphs [18–20]. The consistency around the cube for two-dimensional quad-graph equations is an analogue of the Yang–Baxter equation—variables on the left and right-hand sides of two hexagons of figure 3 are the same. The consistency in this paper implies the different variables on the left and right-hand sides; it defines a map. Therefore, it is an analogue of the *local Yang–Baxter* equation or the *tetrahedral Zamolodchikov algebra* [21–23], see the following section.

Equations (32) and (35) express the discrete spacetime fields $\{\mathcal{A}_v(x,y,t+1) : v = a,b,c; x,y \in \mathbb{Z}\}$ in terms of the fields $\{\mathcal{A}_v(x,y,t) : v = a,b,c; x,y \in \mathbb{Z}\}$. The space-like discrete surface is the collection of hexagons of figure 3 for all $x, y \in \mathbb{Z}$ and fixed t. This is the *honeycomb lattice*. Equations (32) and (35) therefore define the discrete-time evolutions of the fields situated at the faces of the honeycomb lattices.

Often in the literature the dual lattices are used. The fields are associated with the edges of dual three-dimensional lattice and the vertices of its section—dual two-dimensional lattice. The lattice dual to the honeycomb one is called the *kagome lattice*.

5. Linear problem as the zero-curvature representation

Equations (32) and (35) are identically equivalent to equations (22) and (25) correspondingly, but look more complicated since we reverse the time direction of $\mathbf{e}_2$ in equation (28). Our way to introduce time and space-like coordinates (15) and (28) is not yet motivated.

Equations (32) define the map of variables $\mathcal{A}_v = (A_v, A_v^*)$ to $\overline{\mathcal{A}}_v = (\overline{A}_v, \overline{A}_v^*)$, $v = a,b,c$. Analogously, equation (35) defines the map of variables $\mathcal{A}_v = (u_v, w_v)$ to $\overline{\mathcal{A}}_v = (\overline{u}_v, \overline{w}_v)$, $v = a,b,c$. Denote both these maps by the symbol $\mathcal{R}_{abc}$. Formally, $\mathcal{R}_{abc}$ is the operator acting in the space of functions of $\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c$:

$$\forall \Phi = \Phi(\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c) \quad : \quad (\mathcal{R}_{abc} \circ \Phi)(\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c) \stackrel{\text{def}}{=} \Phi(\overline{\mathcal{A}}_a, \overline{\mathcal{A}}_b, \overline{\mathcal{A}}_c). \tag{36}$$

The point is that the maps (32) and (35) are two basic solutions of the *functional* (set theoretical) tetrahedron equation,

$$\mathcal{R}_{abc}\mathcal{R}_{ade}\mathcal{R}_{bdf}\mathcal{R}_{cef} = \mathcal{R}_{cef}\mathcal{R}_{bdf}\mathcal{R}_{ade}\mathcal{R}_{abc}. \tag{37}$$

It may be verified straightforwardly, see the appendix.

Take up now the question: why the maps (32) and (35) satisfy the functional tetrahedron equation. The reason is that the auxiliary linear problems defining the maps of dynamical variables from the left-hand sides of figures 3 and 4 to the right-hand sides, being equivalent to (21) and (23), are the *correctly oriented* zero-curvature representations of three-dimensional integrable models in discrete spacetime.

Let me demonstrate this statement for the map (32). The linear equations for the single face, see figure 1,

$$\psi_{1,\mathbf{n}+\mathbf{e}_2} = \psi_{1,\mathbf{n}} + A_a^*\psi_{2,\mathbf{n}}, \qquad \psi_{2,\mathbf{n}+\mathbf{e}_1} = \psi_{2,\mathbf{n}} + A_a\psi_{1,\mathbf{n}}, \tag{38}$$

may be rewritten in the matrix form as

$$\begin{pmatrix}\psi_{1,\mathbf{n}}\\ \psi_{2,\mathbf{n}+\mathbf{e}_1}\end{pmatrix} = X[\mathcal{A}_a]\cdot\begin{pmatrix}\psi_{1,\mathbf{n}+\mathbf{e}_2}\\ \psi_{2,\mathbf{n}}\end{pmatrix}, \qquad X[\mathcal{A}_a] = \begin{pmatrix}1 & -A_a^*\\ A_a & 1-A_aA_a^*\end{pmatrix}. \tag{39}$$

The linear equations for all three faces of the left-hand side of figure 3 may be written as

$$\begin{gathered}\begin{pmatrix}\psi_1\\ \psi_{2,1}\end{pmatrix} = X[\mathcal{A}_a]\begin{pmatrix}\psi_{1,2}\\ \psi_1\end{pmatrix}, \qquad \begin{pmatrix}\psi_{1,2}\\ \psi_{3,12}\end{pmatrix} = X[\mathcal{A}_b]\begin{pmatrix}\psi_{1,23}\\ \psi_{3,2}\end{pmatrix},\\ \begin{pmatrix}\psi_2\\ \psi_{3,2}\end{pmatrix} = X[\mathcal{A}_c]\begin{pmatrix}\psi_{2,3}\\ \psi_3\end{pmatrix}\end{gathered} \tag{40}$$

with the same matrix function $X[\mathcal{A}]$ (39). Iterating these matrix equations, we come to

$$\begin{pmatrix}\psi_1\\ \psi_{2,1}\\ \psi_{3,12}\end{pmatrix} = X_{12}[\mathcal{A}_a]X_{13}[\mathcal{A}_b]X_{23}[\mathcal{A}_c]\begin{pmatrix}\psi_{1,23}\\ \psi_{2,3}\\ \psi_3\end{pmatrix}, \tag{41}$$

where $X_{ij}[\mathcal{A}_v]$ are the three-by-three block-diagonal matrices, X_{ij} coincides with (39) in (ij)-block and has the unity in complimentary block. For instance,

$$X_{12}[\mathcal{A}_a] = \begin{pmatrix}1 & -A_a^* & 0\\ A_a & 1-A_aA_a^* & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{42}$$

Analogously, the right-hand side of figure 3 provides

$$\begin{pmatrix}\psi_1\\ \psi_{2,1}\\ \psi_{3,12}\end{pmatrix} = X_{23}[\overline{\mathcal{A}}_a]X_{13}[\overline{\mathcal{A}}_b]X_{12}[\overline{\mathcal{A}}_c]\begin{pmatrix}\psi_{1,23}\\ \psi_{2,3}\\ \psi_3\end{pmatrix}. \tag{43}$$

Thus, the consistency of the linear problem around the cube is the *Korepanov* equation [24, 22]

$$X_{12}[\mathcal{A}_a]X_{13}[\mathcal{A}_b]X_{23}[\mathcal{A}_c] = X_{23}[\overline{\mathcal{A}}_c]X_{13}[\overline{\mathcal{A}}_b]X_{12}[\overline{\mathcal{A}}_a]. \tag{44}$$

Using definition (36), we may rewrite the Korepanov equation in the form similar to tetrahedral Zamolodchikov algebra [23],

$$X_{12}[\mathcal{A}_a]X_{13}[\mathcal{A}_b]X_{23}[\mathcal{A}_c] = \mathcal{R}_{abc}\circ X_{23}[\mathcal{A}_c]X_{13}[\mathcal{A}_b]X_{12}[\mathcal{A}_a]. \tag{45}$$

Tetrahedron equation (37) is the equivalence of decompositions of the uniquely defined map $\mathcal{A}_v \to \overline{\overline{\mathcal{A}}}_v$

$$X_{12}[\mathcal{A}_a]X_{13}[\mathcal{A}_b]X_{23}[\mathcal{A}_c]X_{14}[\mathcal{A}_d]X_{24}[\mathcal{A}_e]X_{34}[\mathcal{A}_f]$$
$$= X_{34}[\overline{\overline{\mathcal{A}}}_f]X_{24}[\overline{\overline{\mathcal{A}}}_e]X_{23}[\overline{\overline{\mathcal{A}}}_c]X_{14}[\overline{\overline{\mathcal{A}}}_d]X_{13}[\overline{\overline{\mathcal{A}}}_b]X_{12}[\overline{\overline{\mathcal{A}}}_a] \tag{46}$$

into two different sequences of elementary maps.

The main difference between (44) and the local Yang–Baxter equation [21, 22] is that the numerical indices in (44) and (45) correspond to the components of the *tensor sum* of one-dimensional vector spaces ψ_i. The Korepanov equation comes from the linear problem directly; therefore, it is genuine multi-dimensional generalization of the Lax representation.

There is no analogous matrix form for the zero-curvature representation of the modified three-wave equations. One has to work directly with the sets of linear relations (23). Nevertheless, the orientation of the faces in figure 4 is correct, and a set-of-linear-relations analysis similar to equation (46) provides the set-theoretical proof of the corresponding tetrahedron equation [25, 26].

6. Poisson brackets for the fundamental maps

The maps (32) and (35) define the discrete-time evolution on the honeycomb lattice. Evolution is the Hamiltonian one if it preserves Poisson brackets.

One may verify it for the map (35), if

$$\{u_v, w_v\} = u_v w_v, \qquad v = a, b, c \tag{47}$$

and any other type bracket for u_v, w_v is zero, then

$$\{\overline{u}_v, \overline{w}_v\} = \overline{u}_v \overline{w}_v, \qquad v = a, b, c \tag{48}$$

and any other type bracket for $\overline{u}_v, \overline{w}_v$ is zero. Therefore (35) is the canonical map [25]. Restoring the space structure according to (33) and (34), we come to the whole system of same-time Poisson brackets conserved by the evolution:

$$\{u_v(x, y, t), w_{v'}(x', y', t)\} = \delta_{v,v'}\delta_{x,x'}\delta_{y,y'} u_v(x, y, t) w_v(x, y, t), \tag{49}$$

any other type same-time bracket is zero. Poisson algebra with such a structure of delta symbols is called the *ultra-local* one.

The Poisson brackets for the evolution (32) are not ultra-local. To get the ultra-locality, we need to modify the discrete linear problem (39). Let there

$$X[\mathcal{A}] = \begin{pmatrix} K & -A^* \\ A & K \end{pmatrix}, \qquad \mathcal{A} = (A, A^*, K), \qquad K^2 = 1 - AA^*. \tag{50}$$

The modified map comes from the Korepanov equation (44):

$$\begin{cases} \overline{A}_a^* = \overline{K}_b^{-1}(K_c A_a^* + K_a A_b^* A_c), & \overline{A}_a = \overline{K}_b^{-1}(K_c A_a + K_a A_b A_c^*), \\ \overline{A}_b^* = K_a K_c A_b^* - A_a^* A_c^*, & \overline{A}_b = K_a K_c A_b - A_a A_c, \\ \overline{A}_c^* = \overline{K}_b^{-1}(K_a A_c^* + K_c A_a A_b^*), & \overline{A}_c = \overline{K}_b^{-1}(K_a A_c + K_c A_a^* A_b). \end{cases} \tag{51}$$

By definition (50) $\overline{K}_v^2 = 1 - \overline{A}_v \overline{A}_v^*$. Additional property of this map is

$$\overline{K}_a \overline{K}_b = K_a K_b, \qquad \overline{K}_b \overline{K}_c = K_b K_c. \tag{52}$$

Map (51) satisfies the functional tetrahedron equation and preserves the ultra-local Poisson brackets [14]

$$\{A_v^*, A_v\} = K_v^2, \qquad \{K_v, A_v\} = -\tfrac{1}{2} K_v A_v, \qquad \{A_v^*, K_v\} = -\tfrac{1}{2} K_v A_v^* \tag{53}$$

any other type bracket is zero.

The principal advantage of the Poisson structure is that it allows one to define the lattice actions with the help of functions generating the canonical transformations. This subject is technically complicated; we postpone it for future publications.

7. Quantization

The Poisson algebra (53) is the quasi-classical limit of q-oscillator algebra [27]

$$\mathbf{A}\mathbf{A}^{\dagger} = 1 - q^{2\mathbf{N}+2}, \qquad \mathbf{A}^{\dagger}\mathbf{A} = 1 - q^{2\mathbf{N}}, \qquad \mathbf{A}q^{\mathbf{N}} = q^{\mathbf{N}+1}\mathbf{A}, \qquad \mathbf{A}^{\dagger}q^{\mathbf{N}} = q^{\mathbf{N}-1}\mathbf{A}^{\dagger}. \qquad (54)$$

The Poisson bracket is the limit of commutator when $q^2 = \mathrm{e}^{-\hbar} \to 1$, $\mathbf{A} \to A$, $\mathbf{A}^{\dagger} \to A^*$ and $q^{\mathbf{N}} \to K$.

The Poisson algebra $\{u, w\} = uw$ (47) is the quasi-classical limit of the Weyl algebra [28]

$$\mathbf{u}\mathbf{w} = q^2\mathbf{w}\mathbf{u} \qquad (55)$$

with $\mathbf{u} \to u$, $\mathbf{w} \to w$ when $q \to 1$.

The whole algebra of observables, corresponding to the set of classical fields $\mathcal{A}_v(x, y)$ is thus the tensor power of the local q-oscillator algebra $\mathcal{A} = (\mathbf{A}, \mathbf{A}^{\dagger}, q^{\mathbf{N}})$ for the quantized three-wave system; and the tensor power of the local Weyl algebra $\mathcal{A} = (\mathbf{u}, \mathbf{w})$ for the modified three-wave system. After the quantization, the indices a, b, c of the operators (or, more generally, the indices are (v, x, y), $v = a, b, c$, $x, y \in \mathbb{Z}$) stand for components of the tensor power.

Quantum maps follow from the quantum linear problems. The quantized version of (39) is [29]

$$\begin{aligned} |\psi_{1,\mathbf{n}}\rangle &= \lambda_a q^{\mathbf{N}_a}|\psi_{1,\mathbf{n}+\mathbf{e}_2}\rangle - \mathbf{A}^{\dagger}{}_a|\psi_{2,\mathbf{n}}\rangle, \\ |\psi_{2,\mathbf{n}+\mathbf{e}_1}\rangle &= q^{-1}\lambda_a\mu_a\mathbf{A}_a|\psi_{1,\mathbf{n}+\mathbf{e}_2}\rangle + \mu_a q^{\mathbf{N}_a}|\psi_{2,\mathbf{n}}\rangle \end{aligned} \quad \Leftrightarrow \quad X[\mathcal{A}_a] = \begin{pmatrix} \lambda_a q^{\mathbf{N}_a} & -\mathbf{A}^{\dagger}{}_a \\ q^{-1}\lambda_a\mu_a\mathbf{A}_a & \mu_a q^{\mathbf{N}_a} \end{pmatrix}. \qquad (56)$$

Here we replace the linear variables ψ_i by vectors $|\psi_i\rangle$ from a formal right module of the whole algebra of observables. Parameters λ_a, μ_a are $\mathbb{C}$-valued spectral parameters; we introduce them for the sake of completeness. The consistency of linear problem of figure 3 (equivalent to the quantum Korepanov equation (44)) may be solved with non-commutative coefficients; the answer is [14, 29]

$$\begin{cases} \overline{\mathbf{A}^{\dagger}}_a = \dfrac{\lambda_c}{\lambda_b} q^{-\overline{\mathbf{N}}_b}\left(q^{\mathbf{N}_c}\mathbf{A}^{\dagger}{}_a + \dfrac{\lambda_a\mu_c}{q} q^{\mathbf{N}_a}\mathbf{A}^{\dagger}{}_b\mathbf{A}_c\right), & \overline{\mathbf{A}}_a = \dfrac{\lambda_b}{\lambda_c} q^{-\overline{\mathbf{N}}_b}\left(q^{\mathbf{N}_c}\mathbf{A}_a + \dfrac{q}{\lambda_a\mu_c} q^{\mathbf{N}_a}\mathbf{A}_b\mathbf{A}^{\dagger}{}_c\right), \\ \overline{\mathbf{A}^{\dagger}}_b = \lambda_a\mu_c q^{\mathbf{N}_a+\mathbf{N}_c}\mathbf{A}^{\dagger}{}_b - \mathbf{A}^{\dagger}{}_a\mathbf{A}^{\dagger}{}_c, & \overline{\mathbf{A}}_b = \dfrac{q^2}{\lambda_a\mu_c} q^{\mathbf{N}_a+\mathbf{N}_c}\mathbf{A}_b - \mathbf{A}_a\mathbf{A}_c, \\ \overline{\mathbf{A}^{\dagger}}_c = \dfrac{\mu_a}{\mu_b} q^{-\overline{\mathbf{N}}_b}\left(q^{\mathbf{N}_a}\mathbf{A}^{\dagger}{}_c + \dfrac{\lambda_a\mu_c}{q} q^{\mathbf{N}_c}\mathbf{A}_a\mathbf{A}^{\dagger}{}_b\right), & \overline{\mathbf{A}}_c = \dfrac{\mu_b}{\mu_a} q^{-\overline{\mathbf{N}}_b}\left(q^{\mathbf{N}_a}\mathbf{A}_c + \dfrac{q}{\lambda_a\mu_c} q^{\mathbf{N}_c}\mathbf{A}^{\dagger}{}_a\mathbf{A}_b\right). \end{cases} \qquad (57)$$

Here $q^{2\overline{\mathbf{N}}_b} = 1 - \overline{\mathbf{A}^{\dagger}}_b\overline{\mathbf{A}}_b$, in addition

$$q^{\overline{\mathbf{N}}_a+\overline{\mathbf{N}}_b} = q^{\mathbf{N}_a+\mathbf{N}_b}, \qquad q^{\overline{\mathbf{N}}_b+\overline{\mathbf{N}}_c} = q^{\mathbf{N}_b+\mathbf{N}_c}, \qquad (58)$$

One may verify that the map (57) is the *automorphism* of the tensor cube of q-oscillator algebra (54).

The quantized version of the linear equation (23) for the modified three-wave system is [26]

$$\varkappa_a|\phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}\rangle - q\mathbf{w}_a|\phi_{\mathbf{n}+\mathbf{e}_1}\rangle - \mathbf{u}_a|\phi_{\mathbf{n}+\mathbf{e}_2}\rangle + \mathbf{u}_a\mathbf{w}_a|\phi_{\mathbf{n}}\rangle = 0. \qquad (59)$$

Here $\varkappa_a$ is a $\mathbb{C}$-valued spectral parameter. The solution of the consistency condition, see figure 4, gives

$$\begin{cases} \overline{\mathbf{u}}_a = \Lambda_2 \mathbf{w}_c^{-1}, & \overline{\mathbf{u}}_b = \Lambda_1 \mathbf{u}_c, & \overline{\mathbf{u}}_c = \mathbf{u}_b \Lambda_1^{-1}, \\ \overline{\mathbf{w}}_a = \mathbf{w}_b \Lambda_3^{-1}, & \overline{\mathbf{w}}_b = \Lambda_3 \mathbf{w}_a, & \overline{\mathbf{w}}_c = \Lambda_2 \mathbf{u}_a^{-1}, \end{cases} \tag{60}$$

where

$$\begin{aligned} \Lambda_1 &= \mathbf{u}_a \mathbf{u}_c^{-1} - q \mathbf{u}_a \mathbf{w}_a^{-1} + \varkappa_a \mathbf{u}_b \mathbf{w}_a^{-1}, \\ \Lambda_2 &= \frac{\varkappa_a}{\varkappa_b} \mathbf{u}_b \mathbf{w}_c + \frac{\varkappa_c}{\varkappa_b} \mathbf{u}_a \mathbf{w}_b - q^{-1} \frac{\varkappa_a \varkappa_c}{\varkappa_b} \mathbf{u}_b \mathbf{w}_b, \\ \Lambda_3 &= \mathbf{w}_a^{-1} \mathbf{w}_c - q \mathbf{u}_c^{-1} \mathbf{w}_c + \varkappa_c \mathbf{u}_c^{-1} \mathbf{w}_b. \end{aligned} \tag{61}$$

One may verify that the map (60) is the automorphism of the tensor cube of the Weyl algebra (55).

Both automorphisms (57) and (60) satisfy the 'functional' tetrahedron equation. In irreducible representations they are the internal automorphisms,

$$\mathcal{R}_{abc} \circ \Phi \equiv R_{abc} \Phi R_{abc}^{-1}. \tag{62}$$

The corresponding operators R_{abc} satisfy the quantum (operator-valued) tetrahedron equations. Matrix elements of R_{abc} are functions of the spectral parameters λ_v, μ_v for (57) and $\varkappa_v$ for (60).

The local maps (57) and (60) define the evolution map on the honeycomb lattice via the identification

$$\begin{array}{lcl} \mathcal{A}_a = \mathcal{A}_{a,x,y}(t) & & \overline{\mathcal{A}}_a = \mathcal{A}_{a,x,y-1}(t+1) \\ \mathcal{A}_b = \mathcal{A}_{b,x,y}(t) & \to & \overline{\mathcal{A}}_b = \mathcal{A}_{b,x,y}(t+1) \\ \mathcal{A}_c = \mathcal{A}_{c,x,y}(t) & & \overline{\mathcal{A}}_c = \mathcal{A}_{c,x-1,y}(t+1) \end{array} \tag{63}$$

in accordance with (30), (31) and (33), (34). The evolution map is the automorphism of the whole algebra of observables. Parameters λ_v, μ_v for the q-oscillator model and $\varkappa_v$ for the Weyl algebra model should be (x, y) independent. Then, in proper representations, the evolution is the internal automorphism given by an evolution operator,

$$\Phi(t+1) = U \Phi(t) U^{-1}. \tag{64}$$

Matrix elements of U may be constructed with the help of matrix elements of local R_{abc}.

Examples of irreducible representations providing a 'good' quantum mechanics are the following. For the q-oscillator algebra it is the case of real $q^2 = \mathrm{e}^{-\hbar}, 0 < q < 1$, and the Fock space: the Fock vacuum is defined by $\mathbf{A}|0\rangle = \mathbf{N}|0\rangle = 0$. The dagger of $\mathbf{A}^\dagger$ stands for the Hermitian conjugation, $\mathbf{N}^\dagger = \mathbf{N}$. If in addition $\frac{\lambda_c}{\lambda_b}$, $\frac{\mu_a}{\mu_b}$ and $\frac{\lambda_a \mu_c}{q}$ in (57) are unitary parameters, then R_{abc} and U are the well-defined unitary operators. Matrix elements of R_{abc} are given in [14]. Evaluation (4) of U^N in the framework of coherent states gives the Feynman-type integral in the discrete spacetime. In the quasi-classical limit $\hbar \to 0$ $(q^2 = \mathrm{e}^{-\hbar})$, it may be shown as

$$\langle A^*_{\mathrm{out}} | U^N | A_{\mathrm{in}} \rangle = \int \mathcal{D}A \mathcal{D}A^* \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S[A, A^*]}, \tag{65}$$

where $S[A, A^*]$ is the lattice action mentioned in the previous section, and the measure of integration is as well the lattice one.

The proper quantum-mechanical representation of the Weyl algebra is either unitary finite-dimensional representations at 'root of unity' $q^{2N} = 1$ or the dual modular representation [30].

At the root of unity the finite-dimensional matrix R_{abc} is the R-matrix for Zamolodchikov–Bazhanov–Baxter model [12] and its high genus extensions [13, 31]. For the modular representation, in addition to the given local Weyl pairs

$$\mathbf{u} = \mathrm{e}^{P}, \qquad \mathbf{w} = \mathrm{e}^{Q}, \qquad [Q, P] = \mathrm{i}\hbar \Rightarrow q^2 = \mathrm{e}^{-\mathrm{i}\hbar}, \tag{66}$$

it is necessary to consider the dual local pairs

$$\mathbf{u}' = \mathrm{e}^{\frac{2\pi}{\hbar}P}, \qquad \mathbf{w}' = \mathrm{e}^{\frac{2\pi}{\hbar}Q}, \qquad q'^2 = \mathrm{e}^{-\mathrm{i}\frac{(2\pi)^2}{\hbar}}. \tag{67}$$

The sets of equations (60) and similar equations for dual pairs define the kernel of R_{abc} unambiguously. In this case R_{abc} and U are well-defined unitary operators. The PQ symbol of U^N in the quasi-classical limit $\hbar \to 0$ is

$$\langle P_{\mathrm{out}}|U^N|Q_{\mathrm{in}}\rangle = \int \mathscr{D}P\mathscr{D}Q\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[P,Q]}, \tag{68}$$

where the measure and the action are the lattice ones.

8. Conclusion

The honeycomb lattice evolution map (63) defined with the help of local automorphisms (57) end (60), and the proper choice of the Hilbert space define unambiguously the evolution operator for the lattice approximation of the corresponding quantum field theory. These quantum field theories are integrable since the underlying quantum auxiliary linear problems (56) and (59) provide the existence of the complete set of the integrals of motion [24, 29, 32–34].

The one-step evolution (63) is the discrete form of the Hamiltonian flow (16) and (17). The quantum lattice Hamiltonian $\mathbf{H}$ is defined by

$$U = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathbf{H}\Delta t}, \tag{69}$$

where $\Delta t = \Delta x = \Delta y$ is the lattice spacing parameter. In the continuous $\Delta t \to 0$ classical $q \to 1$ limit the quantum lattice Hamiltonian becomes exactly (16). However, on the lattice Δt is finite (in this paper we used the scale $\Delta t = 1$), and therefore $\mathbf{H}$ is not a polynomial in the algebra of observables. The evolution operator is the principal object of the field theory rather than a Hamiltonian.

The fundamental problem of the field theory is the calculation of the spectrum of evolution operator. This is the open question in three-dimensional models. Spectral equations for the evolution operators are not yet known (except for some 3d $\to$ 2d limits, e.g. [35] for the Weyl algebra and [36] for the q-oscillator algebra).

Let me conclude the paper with a discussion of the role of simplex equations in the quantum field theory. The linear problem is the starting point of the integrability. The tetrahedron equation (37) and the Yang–Baxter equation for the two-dimensional models are the elementary consequences of linear problem as the zero-curvature representation. Nevertheless, the simplex configurations may be considered as fragments of the spacetime lattice, for instance

$$Z = \int \underbrace{\mathrm{d}\phi_1'\,\mathrm{d}\phi_2'\,\mathrm{d}\phi_3'}_{\mathscr{D}\phi}\cdots \underbrace{\langle\phi_1\phi_2|R_{12}|\phi_1',\phi_2'\rangle\langle\phi_1',\phi_3|R_{13}|\phi_1'',\phi_3'\rangle\langle\phi_2',\phi_3'|R_{23}|\phi_2'',\phi_3''\rangle}_{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\phi]}}\cdots. \tag{70}$$

Here for brevity we consider a two-dimensional theory and the triangle configuration. Such a partition function corresponds to the Feynman integral (1) with the discrete measure definition (2) and some particular discretization. The Yang–Baxter equation (and the D-simplex equations in general) is the condition of Z-invariance: the partition function (70) does not depend on particular details of the discretization, it is an invariant function of the boundary fields only. Thus the natural role of simplex equations in the lattice field theory is the conditions of self-consistent definition of the measure (2).

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Appendix. Verification of the functional tetrahedron equations

The functional tetrahedron equation for the map (32) may be verified with the help of Maple 9.5 routine:

```
restart;

R:=proc(a,b,c,var) # x stands for A, y stands for A^*
   local X,Y;
   X[2]:= x[b]*(1-x[a]*y[a])-x[a]*x[c];
   Y[2]:= y[b]*(1-x[c]*y[c])-y[a]*y[c];
   X[1]:= (x[a]*(1-x[c]*y[c])+x[b]*y[c]*(1-x[a]*y[a]))/(1-X[2]*Y[2]);
   Y[1]:= y[a]+y[b]*x[c]; X[3]:= x[c]+y[a]*x[b];
   Y[3]:= (y[c]*(1-x[a]*y[a])+x[a]*y[b]*(1-x[c]*y[c]))/(1-X[2]*Y[2]);
   simplify(subs([x[a]=X[1],y[a]=Y[1],x[b]=X[2],y[b]=Y[2],x[c]=X[3],
   y[c]=Y[3]],var))
end;

TE:= var->R(1,2,3,R(1,4,5,R(2,4,6,R(3,5,6,var))))
        -R(3,5,6,R(2,4,6,R(1,4,5,R(1,2,3,var))));

for k from 1 to 6 do TE(x[k]); TE(y[k]); od;
```

Verification of the tetrahedron equation for the map (35) is

```
restart;

R:= proc(a,b,c,var)
   local U,W;
   U[1]:= (u[a]*w[b]+u[b]*w[c]-u[b]*w[b])/w[c];
   W[1]:= w[a]*w[b]*u[c]/(u[c]*w[c]+w[a]*w[b]-w[a]*w[c]);
   U[2]:= (u[a]*w[a]+u[b]*u[c]-u[a]*u[c])/w[a];
   W[2]:= (u[c]*w[c]+w[a]*w[b]-w[a]*w[c])/u[c];
   U[3]:= w[a]*u[b]*u[c]/(u[a]*w[a]+u[b]*u[c]-u[a]*u[c]);
   W[3]:= (u[a]*w[b]+u[b]*w[c]-u[b]*w[b])/u[a];
   simplify(subs([u[a]=U[1],w[a]=W[1],u[b]=U[2],w[b]=W[2],u[c]=U[3],
   w[c]=W[3]],var))
end;

TE:= var->R(1,2,3,R(1,4,5,R(2,4,6,R(3,5,6,var))))
        -R(3,5,6,R(2,4,6,R(1,4,5,R(1,2,3,var))));

for k from 1 to 6 do TE(u[k]); TE(w[k]); od;
```

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